Solucionario Examen Ordinario de Cálculo Integral
Academia Enero-Junio 2019

Problema 1.

For $0 \le t \le 10$, $b(t) > d(t)$, so the area between the curves is given by

$$\int_0^{10} [b(t) - d(t)]\, dt = \int_0^{10} (2200e^{0.024t} - 1460e^{0.018t})\, dt = \left[\frac{2200}{0.024}e^{0.024t} - \frac{1460}{0.018}e^{0.018t}\right]_0^{10}$$

$$= \left(\frac{275{,}000}{3}e^{0.24} - \frac{730{,}000}{9}e^{0.18}\right) - \left(\frac{275{,}000}{3} - \frac{730{,}000}{9}\right) \approx 8868 \text{ people}$$

This area A represents the increase in population over a 10-year period.

Problema 2.

$x^2 + y^2 = r^2 \quad \Leftrightarrow \quad x^2 = r^2 - y^2$

$$V = \pi \int_{r-h}^{r} (r^2 - y^2)\, dy = \pi \left[r^2 y - \frac{y^3}{3}\right]_{r-h}^{r} = \pi \left\{\left[r^3 - \frac{r^3}{3}\right] - \left[r^2(r-h) - \frac{(r-h)^3}{3}\right]\right\}$$

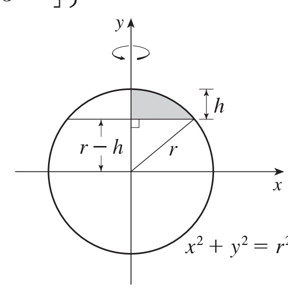
$$= \pi\left\{\tfrac{2}{3}r^3 - \tfrac{1}{3}(r-h)\left[3r^2 - (r-h)^2\right]\right\}$$

$$= \tfrac{1}{3}\pi\left\{2r^3 - (r-h)\left[3r^2 - \left(r^2 - 2rh + h^2\right)\right]\right\}$$

$$= \tfrac{1}{3}\pi\left\{2r^3 - (r-h)\left[2r^2 + 2rh - h^2\right]\right\}$$

$$= \tfrac{1}{3}\pi\left(2r^3 - 2r^3 - 2r^2h + rh^2 + 2r^2h + 2rh^2 - h^3\right)$$

$$= \tfrac{1}{3}\pi\left(3rh^2 - h^3\right) = \tfrac{1}{3}\pi h^2(3r - h), \text{ or, equivalently, } \pi h^2\left(r - \frac{h}{3}\right)$$



Problema 3.
F(2)=0
F´(2)=4e^2
F”(2)=6e^2

Problema 4.

The prey hits the ground when $y = 0 \quad \Leftrightarrow \quad 180 - \frac{1}{45}x^2 = 0 \quad \Leftrightarrow \quad x^2 = 45 \cdot 180 \quad \Rightarrow \quad x = \sqrt{8100} = 90$,

since x must be positive. $y' = -\frac{2}{45}x \quad \Rightarrow \quad 1 + (y')^2 = 1 + \frac{4}{45^2}x^2$, so the distance traveled by the prey is

$$L = \int_0^{90} \sqrt{1 + \frac{4}{45^2}x^2}\, dx = \int_0^4 \sqrt{1 + u^2}\left(\tfrac{45}{2}\, du\right) \qquad \begin{bmatrix} u = \frac{2}{45}x, \\ du = \frac{2}{45}\, dx \end{bmatrix}$$

$$\overset{21}{=} \tfrac{45}{2}\left[\tfrac{1}{2}u\sqrt{1+u^2} + \tfrac{1}{2}\ln\left(u + \sqrt{1+u^2}\right)\right]_0^4 = \tfrac{45}{2}\left[2\sqrt{17} + \tfrac{1}{2}\ln\left(4 + \sqrt{17}\right)\right] = 45\sqrt{17} + \tfrac{45}{4}\ln\left(4 + \sqrt{17}\right) \approx 209.1 \text{ m}$$

Problema 5.

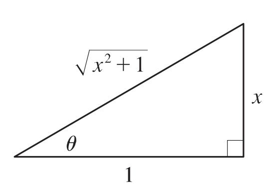
a)

Let $x = \tan\theta$, so that $dx = \sec^2\theta\, d\theta$. Then

$$\int \frac{dx}{x\sqrt{x^2+1}} = \int \frac{\sec^2\theta\, d\theta}{\tan\theta\, \sec\theta} = \int \frac{\sec\theta}{\tan\theta}\, d\theta$$

$$= \int \csc\theta\, d\theta = \ln|\csc\theta - \cot\theta| + C$$

$$= \ln\left|\frac{\sqrt{x^2+1}}{x} - \frac{1}{x}\right| + C = \ln\left|\frac{\sqrt{x^2+1} - 1}{x}\right| + C$$



b)

$$\frac{3x^3 - x^2 + 6x - 4}{(x^2+1)(x^2+2)} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{x^2+2} \quad \Rightarrow \quad 3x^3 - x^2 + 6x - 4 = (Ax+B)(x^2+2) + (Cx+D)(x^2+1).$$

Equating the coefficients gives $A + C = 3$, $B + D = -1$, $2A + C = 6$, and $2B + D = -4 \quad \Rightarrow$

$A = 3$, $C = 0$, $B = -3$, and $D = 2$. Now

$$\int \frac{3x^3 - x^2 + 6x - 4}{(x^2+1)(x^2+2)}\,dx = 3\int \frac{x-1}{x^2+1}\,dx + 2\int \frac{dx}{x^2+2} = \frac{3}{2}\ln(x^2+1) - 3\tan^{-1} x + \sqrt{2}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right) + C.$$

c)

$$\int u\,dv = uv - \int v\,du$$

$$\int \sec^3 x\,dx = \sec x \tan x - \int \sec x \tan^2 x\,dx$$

$$\int \sec^3 x\,dx = \sec x \tan x - \int \sec x(\sec^2 x - 1)\,dx$$

$$\int \sec^3 x\,dx = \sec x \tan x - \int \sec^3 x\,dx + \int \sec x\,dx$$

$$2\int \sec^3 x\,dx = \sec x \tan x + \int \sec x\,dx$$

$$2\int \sec^3 x\,dx = \sec x \tan x + \ln|\sec x + \tan x| + C$$

$$\int \sec^3 x\,dx = \frac{1}{2}\sec x \tan x + \frac{1}{2}\ln|\sec x + \tan x| + C.$$

d)

$$\int_1^\infty \frac{1}{(2x+1)^3}\,dx = \lim_{t\to\infty}\int_1^t \frac{1}{(2x+1)^3}\,dx = \lim_{t\to\infty}\int_1^t \tfrac{1}{2}(2x+1)^{-3}\,2\,dx = \lim_{t\to\infty}\left[-\frac{1}{4(2x+1)^2}\right]_1^t$$

$$= -\frac{1}{4}\lim_{t\to\infty}\left[\frac{1}{(2t+1)^2} - \frac{1}{9}\right] = -\frac{1}{4}\left(0 - \frac{1}{9}\right) = \frac{1}{36}$$